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Crossed Products with Respect to Different Kernel Functors

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INTRODUCTION

A classical result in the theory of the Brauer group of a commutative ring is the crossed product theorem, stating that the part of the Brauer group split by a faithfully projective extension S is isomorphic to a second cohomology group, namely, the Amitsur cohomology group $H^2(S/R, U)$, under the condition that the Picard groups of S and $S \otimes S$ are trivial (cf., e.g., [7, 9]). In a previous paper, A. Verschoren and the author generalized this property to the Brauer group relative to an idempotent kernel functor σ on R-mod (cf. [5, 14]). In this paper, we observe that the conditions on the Picard group may be weakened, by introducing a second idempotent kernel functor τ on R-mod, with a stronger associated Gabriel topology. The condition on the Picard group relative to σ is then replaced by one on σ, and the cohomology group $H^2(S/R, U_\sigma)$ by a more general $H^2_\sigma(S/R, U_\tau)$, which will be defined in Section 1.

The condition on Pic (S, τ) is always fulfilled if one takes $\tau = \sigma_p$, for some prime ideal p not contained in the Gabriel filter. Hence Br $(S/R, \sigma)$ may always be written as a second cohomology group; this property also holds for the "absolute" Brauer group: let σ be the trivial idempotent kernel functor.

In Section 2, we apply our method to embed the full (relative) Brauer group in a cohomology group. In the absolute case, the obtained cohomology group may be related to the well-known cohomological Brauer group, using a theorem due to Artin [1].

As another application, a crossed product theorem for the Brauer class group (introduced by M. Orzech in [10]) is given.

Throughout this paper, we make extensive use of the notations and results obtained in [5, 13, 14]. For generalities on torsion theories, we refer to [8] and [12]. We briefly recall some notations: for an idempotent ker-

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nel functor on R-mod, $\mathscr{L}(\sigma)$ denotes the Gabriel filter. $X(\sigma) = \mathrm{Spec}(R) - \mathscr{L}(\sigma)$, and $\mathscr{C}(\sigma)$ consists of the maximal elements in $X(\sigma)$. Q_σ is the associated localization functor, and (R, σ)-mod is the category of σ-closed R-modules. The relative Picard group $\mathrm{Pic}(R, \sigma)$ consists of isomorphism classes of σ-invertible R-modules, i.e., R-modules I such that $I \perp J = Q_\sigma(I \otimes J) \cong Q_\sigma(R)$ for some $J \in R$-mod. $M \in (R, \sigma)$-mod is called a σ-progenerator if M is σ-finitely presented and σ-faithful (cf. [13]) and M_p is R_p-projective for all $p \in \mathscr{C}(\sigma)$.

An R-algebra A is called a σ-Azumaya algebra if A is a σ-progenerator, and $A \perp A^0 \cong \mathrm{End}_R(A)$. The relative Brauer group $\mathrm{Br}(R, \sigma)$ then consists of equivalence classes of σ-Azumaya algebras, where two σ-Azumaya algebras A, B are equivalent if there exist σ-progenerators P, Q such that $A \perp \mathrm{End}_R(P) \cong B \perp \mathrm{End}_R(Q)$.

1. Crossed Products with Respect to Different Kernel Functors

Let R be a commutative ring, and σ, τ some idempotent kernel functors on R-mod such that $\sigma \leqslant \tau$. Also assume that R is σ-Noetherian (i.e., R satisfies the ascending chain condition on σ-closed ideals). Then R is also τ-Noetherian, since every τ-closed R-module is σ-closed. It is easily verified that the following conditions are equivalent: $\sigma \leqslant \tau$; $X(\tau) \subset X(\sigma)$; $\mathscr{T}_\sigma \subset \mathscr{T}_\tau$; $\sigma(M) \subset \tau(M)$, for all $M \in R$-mod; $\mathscr{L}(\sigma) \subset \mathscr{L}(\tau)$; $\mathscr{F}_\tau \subset \mathscr{F}_\sigma$. In this case, every τ-injective module is σ-injective.

Let S be a σ-faithfully flat R-algebra, then [5, 1.6] yields that $Q_\tau(S)$ is a τ-faithfully flat R-algebra. Also, for any σ-progenerator M, $Q_\tau(M)$ is a τ-progenerator [13, III.1.1, III.1.2]. σ, τ induce idempotent kernel functors $\bar{\sigma}, \bar{\tau}$ on S-mod. Recall that

$$X(\bar{\sigma}) = \{P \in \mathrm{Spec}(S) \colon P \cap R \in X(\sigma)\}.$$

Consider the categories $\mathrm{Pic}(S, \bar{\sigma})$, $\mathrm{Pic}(S, \bar{\tau})$ of invertible $\bar{\sigma}$- and $\bar{\tau}$-closed S-modules, with product $\perp_\sigma$ and $\perp_\tau$. Then $Q_{\bar{\tau}}$ induces a cofinal covariant functor $\mathrm{Pic}(S, \bar{\sigma}) \to \mathrm{Pic}(S, \bar{\tau})$, which is product preserving. Using [2], we therefore have an exact sequence

$$K_1 \underline{\underline{\mathrm{Pic}}}(S, \bar{\sigma}) \to K_1 \underline{\underline{\mathrm{Pic}}}(S, \bar{\tau}) \to K_1 \phi \underline{\underline{Q}}_{\bar{\tau}} \to K_0 \underline{\underline{\mathrm{Pic}}}(S, \bar{\sigma}) \to K_0 \underline{\underline{\mathrm{Pic}}}(S, \bar{\tau}),$$

or

$$U_\sigma(S) \to U_\tau(S) \xrightarrow{\ d\ } Pu_{\sigma\tau}(S) \to \mathrm{Pic}_\sigma(S) \to \mathrm{Pic}_\tau(S),$$

where $U_\sigma(S) = U(Q_\sigma(S))$, $\mathrm{Pic}_\sigma(S) = \mathrm{Pic}(S, \bar{\sigma})$, $Pu_{\sigma\tau}(S) = Pu(S, \bar{\sigma}, \bar{\tau}) =$

$K_1 \phi \underline{Q}_{\bar{\tau}}$. The classes in $Pu_{\sigma\tau}(S)$ are represented by elements of the form (C_1, α, C_2), where $C_i \in \underline{\underline{\mathrm{Pic}}}(S, \bar{\sigma})$, α: $Q_{\bar{\tau}}(C_1) \to Q_{\bar{\tau}}(C_2)$ an isomorphism. (C_1, α, C_2) is isomorphic to (C_1', α', C_2') if we have isomorphisms $f_i: C_i \to C_i'$ such that the diagram

$$\begin{array}{ccc} Q_{\bar{\tau}}(C_1) & \xrightarrow{\alpha} & Q_{\bar{\tau}}(C_2) \\ {\scriptstyle Q_{\bar{\tau}}(f_1)}\downarrow & & \downarrow{\scriptstyle Q_{\bar{\tau}}(f_2)} \\ Q_{\bar{\tau}}(C_1') & \xrightarrow{\alpha'} & Q_{\bar{\tau}}(C_2') \end{array}$$

commutes.

In $Pu_{\sigma\tau}(S)$, we have the following relations:

$$[(C_1, \beta\alpha, C_3)] = [(C_1, \alpha, C_2)][(C_2, \beta, C_3)];$$
$$[(C_1, \alpha, C_2)][(C_1', \alpha', C_2')] = [(C_1 \perp C_1', \alpha \perp \alpha', C_2 \perp C_2')].$$

Using these relations, we may find that $[(C, 1, C)]^2 = [(C, 1, C)]$, hence $[(C, 1, C)] = 1$, for all $C \in \underline{\underline{\mathrm{Pic}}}(S, \bar{\sigma})$. It follows that every $x \in Pu_{\sigma\tau}(S)$ may be represented by $(Q_\sigma(S), \alpha, C)$ for some $\alpha \in U_\tau(S)$, $C \in \underline{\underline{\mathrm{Pic}}}(S, \bar{\sigma})$.

It may be checked easily that $Pu_{\sigma\tau}$: $S \to Pu(S, \bar{\sigma}, \bar{\tau}) = \overline{Pu}_{\sigma\tau}(S)$ determines a covariant functor from the category of R-algebras to abelian groups. Also note that $Pu_{\sigma\tau} = Pu_{\sigma\tau} \circ Q_\tau$. We summarize our results

1.1. Proposition. *Let $\sigma \leqslant \tau$ be idempotent kernel functors on* R-mod. *Then for any commutative R-algebra S, we have a natural exact sequence*

$$\begin{aligned} &U_\sigma(S) \to U_\tau(S) \xrightarrow{d} \\ &\qquad Pu_{\sigma\tau}(S) \to \mathrm{Pic}_\sigma(S) \to \mathrm{Pic}_\tau(S) \to \mathrm{Bcl}(S, \bar{\sigma}, \bar{\tau}) \to \mathrm{Br}(S, \bar{\sigma}) \to \mathrm{Br}(S, \bar{\tau}). \end{aligned}$$

Every element of $Pu_{\sigma\tau}(S)$ may be represented by $(Q_\sigma(S), \alpha, C)$, for some $\alpha \in U_\tau(S)$, $C \in \underline{\underline{\mathrm{Pic}}}(S, \bar{\sigma})$. The map d is given by $d(u) = [(Q_\sigma(S), u, Q_\sigma(S))]$.

Proof. The first part of the sequence has been derived above. For the second part, we refer to [13, III.4] and [9].

d determines a mapping between the Amitsur complexes $\mathscr{C}(U_\tau)$ and $\mathscr{C}(Pu_{\sigma\tau})$, namely:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U_\tau(S) & \xrightarrow{\Delta_0} & U_\tau(S^{(2)}) & \xrightarrow{\Delta_1} & U_\tau(S^{(3)}) & \xrightarrow{\Delta_2} & \cdots \\ & & \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_2} & & \\ 1 & \longrightarrow & Pu_{\sigma\tau}(S) & \xrightarrow{D_0} & Pu_{\sigma\tau}(S^{(2)}) & \xrightarrow{D_1} & Pu_{\sigma\tau}(S^{(3)}) & \xrightarrow{D_2} & \cdots. \end{array}$$

A new complex $(C, \nabla) = \mathscr{C}_{\sigma\tau}(U_\tau)$ is defined in the following way (cf. [3]):

$$C^n = U_\tau(S^{(n+1)}) \times Pu_{\sigma\tau}(S^{(n)});$$

$$\nabla_n(u, T) = (\Delta_n u, D_{n-1} T(d_n u)^{-1}).$$

The corresponding cohomology groups are denoted by

$$H^n_\sigma(S/R, U_\tau) = \mathrm{Ker}\nabla_n / \mathrm{Im}\nabla_{n-1}.$$

Using [6, IV.3], we get a long exact sequence

$$\cdots \to H^{n-1}(S/R, Pu_{\sigma\tau}) \to H^n_\sigma(S/R, U_\tau) \to H^n(S/R, U_\tau) \to \cdots \qquad (*)$$

Recall from [5] that, if S is a σ-progenerator, and $\mathrm{Pic}_\sigma(S) = \mathrm{Pic}_\sigma(S \perp S) = 1$, then $\mathrm{Br}(S/R, \sigma) \cong H^2(S/R, U_\sigma)$. We now generalize this property.

1.2. THEOREM. *Let $\sigma \leqslant \tau$ be idempotent kernel functors on R-mod, and suppose that R is σ-Noetherian. Let S be a σ-faithfully flat R-algebra. If $\mathrm{Pic}_\tau(S) = \mathrm{Pic}_\tau(S \perp S) = 1$, then there exists a natural monomorphism β: $\mathrm{Br}(S/R, \sigma) \to H^2_\sigma(S/R, U_\tau)$, which is an isomorphism if S is a σ-progenerator.*

Proof. Let A be a σ-Azumaya algebra split by S, then there exists a: $A \perp_\sigma S \to \mathrm{End}_S(P)$, for some σ-progenerator P. Define ϕ: $\mathrm{End}_{S \perp S}(S \perp_\sigma P) \to \mathrm{End}_{S \perp S}(P \perp_\sigma S)$ as $\phi = (a \perp 1_S) \circ (\theta \perp 1_S) \circ (1_S \perp a)^{-1}$ (where $\theta = Q_\sigma(t)$, t the usual switch map). By the relative version of the Morita theorem, ϕ is induced by f: $P_1 \perp_2 I \to P_2$, for some $I \in \underline{\underline{\mathrm{Pic}}}_\sigma(S \perp_\sigma S)$. By assumption, $[Q_\tau(I)] = 1$ in $\mathrm{Pic}_\tau(S \perp S)$, so there exists an isomorphism α: $S \perp_\tau S \to I$. Consider $x = [(S \perp_\sigma S, \alpha, I)] \in Pu_{\sigma\tau}(S \perp_\sigma S)$. Localizing at τ, we obtain an isomorphism $g = Q_\tau(f) \circ (1 \perp \alpha)$: $S \perp_\tau P \to P \perp_\tau S$. $g_2^{-1} g_3 g_1$ is given by multiplication by a unit $u \in S \perp_\tau S \perp_\tau S$, which is a cocycle, by a classical argument.

We claim that (u, x) represents an element of $H^2_\sigma(S/R, U_\tau)$. Indeed, we have to show that $D_1 x = d_2 u$, or $[(S \perp_\sigma S \perp_\sigma S, u, S \perp_\sigma S \perp_\sigma S)] = [(S \perp_\sigma S \perp_\sigma S, \alpha_2^{-1}\alpha_3\alpha_1, I_2^{-1} \perp_3 I_3 \perp_3 I_1)]$, which is the case, in view of the following commutative diagram of $S \perp_\tau S \perp_\tau S$-isomorphisms:

$$\begin{array}{ccc}
P_{11} & \xrightarrow{\quad u \quad} & P_{11} \\
{\scriptstyle \mathrm{Id}}\downarrow & & \downarrow {\scriptstyle Q_\tau(f_1^{-1} f_3^{-1} f_2)} \\
P_{11} & \xrightarrow{\alpha_2^{-1}\alpha_3\alpha_1} & P_{11} \perp_3 I_2^{-1} \perp_3 I_3 \perp_3 I_1
\end{array}$$

Also observe that $[(P_{11}, 1, P_{11})] = 1$. Let us now prove that $[A] \to [(u, x)]$ defines a homomorphism β: $\mathrm{Br}(S/R, \sigma) \to H^2_\sigma(S/R, U_\tau)$. Let

a': $A\perp S \to \mathrm{End}_S(P')$ be another splitting map for A, then $a'a^{-1}$: $\mathrm{End}_S(P) \to \mathrm{End}_S(P')$ is induced by some b: $P \perp_\sigma J \to P'$ for some $J \in \underline{\underline{\mathrm{Pic}}}_\sigma(S)$. There exists an isomorphism γ: $S \to Q_\tau(J)$, since $\mathrm{Pic}_\tau(S)=1$. Define the map $c = Q_\tau(b) \circ (1\perp\gamma)$: $Q_\tau(P) \to Q_\tau(P')$. Then ϕ' is induced by $f' = c_2 g c_1^{-1}$, hence $u' = (c_2 g c_1^{-1})_2^{-1}\,(c_2 g c_1^{-1})_3\,(c_2 g c_1^{-1})_1 = c_{12} u c_{12}^{-1} = u$. Also $x' = [(S\perp S,\ \gamma_1^{-1}\perp\gamma_2\perp\alpha,\ J_1^{-1}\perp J_2 \perp I)] = [(S\perp S,\ \alpha,\ I)][(S\perp S,\ Q_\tau(\delta),\ J_1^{-1}\perp J_2)]$, where $\gamma_1^{-1}\perp\gamma_2 = Q_\tau(\delta)$, since $[(J_1^{-1}\perp J_2)]=1$ in $\mathrm{Pic}_\tau(S\perp S)$.

Furthermore, if a: $A\perp S \to \mathrm{End}_S(P)$ is a splitting map, then $a' = a\perp 1$: $A\perp \mathrm{End}_R(Q) \perp S \to \mathrm{End}_S(P\perp_1 Q_2)$ is a splitting map for $A\perp\mathrm{End}_R(Q)$. It is easily shown that $[(u,x)] = [(u',x')]$ in $H^2_\sigma(S/R, U_\tau)$. It is also clear that β is a homomorphism.

Next, to show that β is a monomorphism, suppose that $(u,x) = (\Delta_1 v, D_0 y(d_1 v)^{-1})$, where (u,x) is obtained as above. Let $y = [(S, \beta, J)]$, then it follows that $[(S\perp S,\ \beta_1,\ J_1)] = [(S\perp S,\ v^{-1}\alpha\perp\beta_2,\ I\perp J_2)]$, hence there exists an isomorphism b in $(S\perp S, \bar\sigma)$-mod, b: $J_1 \to I\perp_2 J_2$, such that $Q_\tau(b)\,\beta_1 = v\alpha\perp\beta_2$. Consider f': $P_1\perp_2 J_1 \to P_1\perp_2 I\perp_2 J_2 \to P_2\perp_2 J_2$, given by $f' = (f\perp 1)\circ(1\perp b)$. Then $Q_\tau(f_2'^{-1} f_3' f_1') = \Delta_1 v^{-1}\cdot u = 1$. Since $Q_{\bar\tau}$ is an exact functor from $(S\perp S, \bar\sigma)$-mod to $(S\perp S, \bar\tau)$-mod, it follows that $f_2'^{-1} f_3' f_1' = 1$, so f' is a σ-descent datum. So there exists a σ-progenerator Q such that the following diagram commutes in $(S\perp S, \bar\sigma)$-mod:

$$\begin{array}{ccc} S\perp Q\perp S & \longrightarrow & P_1\perp_2 J_1 \\ \downarrow & & \downarrow{\scriptstyle f'} \\ Q\perp S\perp S & \longrightarrow & P_2\perp_2 J_2 \end{array}$$

Taking induced isomorphisms, it follows that $A \cong \mathrm{End}_R(Q)$. Note that f' also induces ϕ.

Finally, suppose that S is a σ-progenerator. We show that β is surjective. Let (u,x) represent an element of $H^2_\sigma(S/R, U_\tau)$, where $x = (S\perp S, \alpha, I)$. Consider the S-module P, which is equal to I^* as an abelian group, and with action defined by $s\cdot z = (s\perp 1)\,z$ (i.e., S acts only on the first factor). Then $[(S\perp_\sigma S\perp_\sigma S,\ \alpha_2^*\perp_3\alpha_3\perp_3\alpha_1,\ I_2^*\perp_3 I_3\perp_3 I_1)] = [(S\perp_\sigma S\perp_\sigma S,\ u,\ S\perp_\sigma S\perp_\sigma S)]$ in $Pu_{\sigma\tau}(S\perp_\sigma S\perp_\sigma S)$, hence there exists a β: $I_2^*\perp_3 I_3\perp_3 I_1 \to S\perp_\sigma S\perp_\sigma S$, such that $Q_\tau(\beta) = u\circ(\alpha_2^{-1}\perp\alpha_3\perp\alpha_1)$. This yields a β': $I_1^*\perp_3 I_3^* \to I_2^*$ with $Q_\tau(\beta') = \alpha_2^{*-1}\circ u\circ(\alpha_1^*\perp\alpha_3^*)$. β' induces an $S\perp_\sigma S$-isomorphism $f = \tau_1\circ\beta'$: $P_1\perp_2 I \to P_2$.

Also $Q_\tau(f_2^{-1} f_3 f_1)$: $Q_\tau(P_{11}) \to Q_\tau(P_{11})$ is just multiplication by u_4 (identifying $Q_\tau(I)$ and $Q_\tau(S\perp S)$ using α, and the cocycle relations on u). Let f induce ϕ: $\mathrm{End}_{S\perp S}(P_1) \to \mathrm{End}_{S\perp S}(P_2)$, then $Q_\tau(\phi_2^{-1}\phi_3\phi_1) = 1$, hence ϕ is a descent datum, by exactness of Q_τ: $(S\perp S, \bar\sigma)$-mod $\to (S\perp S, \bar\tau)$-mod. Consequently ϕ defines a σ-Azumaya algebra A such that $A\perp S \cong \mathrm{End}_S(P)$. It may be checked that $\beta(A) = [(u,x)]$.

1.3. Corollary. *Let $\sigma \leqslant \tau$ be idempotent kernel functors on R*-mod, *and suppose that R is σ-Noetherian. Let S be a σ-Galois extension of R, with group G. If* $\mathrm{Pic}_\tau(S)=1$, then $\mathrm{Br}(S/R, \sigma) \cong H^2_\sigma(G, U_\tau(S))$.

Proof. From the results of [5], it follows that $Q_\tau(S)$ is a τ-Galois extension of R with group G. Then apply 1.2. $H^2_\sigma(G, U_\tau(S))$ is defined similarly to $H^2_\sigma(S/R, U_\tau)$.

2. Applications

Let R be a commutative ring, and σ an idempotent kernel functor on R-mod such that R is σ-noetherian.

A σ-closed R-algebra S is called a *σ-étale covering* of R if $Q_p(S)$ is a σ-étale covering of $Q_p(R)$, for every p in $\mathscr{C}(\sigma)$. If $\sigma \leqslant \tau$, then $Q_\tau(S)$ is a τ-étale covering, if S is a σ-étale covering.

2.1. Theorem. *If R is σ-Noetherian, then there exists a natural embedding β:* $\mathrm{Br}(R, \sigma) \to \varinjlim H^2_\sigma(S/R, U_p)$, *where the limit is taken over all σ-étale coverings of R, p being a fixed element in* $\mathscr{C}(\sigma)$, $U_p = U_{\sigma_p}$.

Proof. $\mathscr{C}(\sigma)$ is a quasicompact space. Using this fact, we may find a σ-étale covering S of R which splits a given σ-Azumaya algebra A: let Q_σ act on a Zariski covering of $\mathscr{C}(\sigma)$. Since $Q_p(S)$ is an étale covering of $Q_p(R)$, $Q_p(S)$ is semi-local, hence $\mathrm{Pic}_\sigma S = \mathrm{Pic}\, Q_p(S) = 1$. So by 1.2, $\mathrm{Br}(S/R, \sigma)$ embeds in $H^2(S/R, U_p)$. Then take the limit over all σ-étale coverings.

2.2. *Note.* If $\sigma = 1$, then we get an embedding $\mathrm{Br}(R) \to^{\beta_p} \varinjlim H^2_1(S/R, U_p)$, for every $p \in \mathrm{Spec}(R)$. Thus $\mathrm{Br}(R)$ may be embedded in a cohomology group without use of Artin's theorem [1]. The connection with the classical embedding β: $\mathrm{Br}(R) \to \varinjlim H^2(S/R, U)$ is given in the next theorem.

2.3. Theorem. *Let R be a Noetherian ring, and* $p \in \mathrm{Spec}(R)$. Then $\varinjlim H^2(S/R, U) \cong \varinjlim H^2_1(S/R, U_{1_p})$.

Proof. First, we prove that there exists an exact sequence

$$\varinjlim H^1(S/R, U_p) \xrightarrow{\beta} \varinjlim H^1(S/R, Pu_{1_p}) \xrightarrow{\gamma} \varinjlim H^2(S/R, U)$$

$$\xrightarrow{\delta} \varinjlim H^2_1(S/R, U_p) \to 1. \qquad (**)$$

Definition of β. Define $\beta([u]) = [d(u)]$.

DEFINITION OF γ. Let $x \in \varinjlim H^1(S/R, Pu_{1_p})$ be represented by $[(S^{(2)}, \alpha, C)] \in H^1(S/R, Pu_{1_p})$. Up to replacing S by an étale covering, we may suppose that $[C]=1$ in $\mathrm{Pic}(S^{(2)})$ (invoking Artin's theorem). By 1.1, x is therefore represented by $[(S^{(2)}, u, S^{(2)})]$, where $u \in U_p(S^{(2)}) = U(Q_p(S^{(2)})$. As $D_1(S^{(2)}, u, S^{(2)}) = (S^{(3)}, u_2^{-1}u_3u_1, S^{(3)})$ is trivial in $Pu_{1_p}(S^{(3)})$, $u_2^{-1}u_3u_1$ is represented by $v \in U(S^{(3)})$, using 1.1. Define $\gamma(x)=[v]$, $\varinjlim H^2(S/R, U)$. It is easily seen that v is a cocycle.

Exactness at $\varinjlim H^1(S/R, Pu_{1_p})$. It is clear that $\gamma \circ \beta = 1$. Suppose that $[v]=\gamma(x)=1$, then there exists an étale covering S' of S such that $v' = w_2^{-1}w_3w_1$ for some $w \in U(S^{(2)})$, where v' is obtained by extension of scalars. Now $d(uw^{-1}) = d(u)$, and $[uw^{-1}] \in H^1(S'/R, U_p)$, so $x = \beta([uw^{-1}])$.

DEFINITION OF δ. Define $\delta([v]) = [(v, 1)]$.

Exactness at $\varinjlim H^2(S/R, U)$. Suppose that $\delta([v])=1$; then there exists an étale covering S' of S such that $[(v', S'^{(3)})]=1$ in $H_1^2(S/R, U_p)$. But then $Q_p(v') = \Delta_1 u$ for some $u \in U(Q_p(S^{(2)}))$, so $v = \gamma(d(u))$.

δ is surjective. Let $[u, x)] \in H_1^2(S/R, U_p)$ represent $a \in \varinjlim H_1^2(S/R, U_p)$. Let $x = [(S^{(2)}, \alpha, I)]$. By Artin's theorem, there exists an étale covering S' of S such that $[I'] = [I \otimes_2 S'^{(2)}]$ is trivial in $\mathrm{Pic}(S'^{(2)})$. Then $x' = [(S'^{(2)}, \alpha', I)] = d_1(v)$ for some $v \in U_p(S'^{(2)})$, and $a = [(u, x)] = [(u', x')] = [(u', d_1 v)] = [(u', d_1 v)][(\Delta_1 v, (d_1 v)^{-1})] = [(u'\Delta_1 v, 1)]$ in $Pu_{1_p}(S^{(3)})$, so $u\Delta_1 v \in U(S^{(3)})$, and $y = \delta([u\Delta_1 v])$.

Now, consider the exact sequence $(*)$ following 1.1. Taking limits over all étale coverings of R, we obtain

$$\varinjlim H^1(S/R, U_p) \to \varinjlim H^1(S/R, Pu_{1_p}) \to \varinjlim H_1^2(S/R, U_p) \to \cdots . \qquad (*).$$

Observe that the maps β in $(*)$ and $(**)$ are identical, and that $\alpha = \delta \circ \gamma$. Indeed, let $x \in \varinjlim H^1(S/R, Pu_{1_p})$ be represented by $[(S^{(2)}, u, S^{(2)})]$, then $\delta \circ \gamma(x) = \delta([\Delta_1 u]) = [(\Delta_1 u, 1)] = \alpha(d_1(u)) = \alpha(x)$. This implies that $\alpha = 1$, hence β is surjective. From $(**)$ it follows that $\gamma = 1$, and δ is injective. Thus δ is an isomorphism.

2.4. *Note.* Observe the analogy with [4, 2.15 and 2.16].

As another application, we give a crossed product theorem for Orzech's Brauer class group (cf. [9]); we denote $\mathrm{Bcl}(S/R, \sigma, \tau) = \mathrm{Ker}(\mathrm{Bcl}(R, \sigma, \tau) \to \mathrm{Bcl}(S, \bar{\sigma}, \bar{\tau}))$.

2.5. COROLLARY. *Let $\sigma \leqslant \tau$ be idempotent kernel functors on R-mod, where R is σ-Noetherian, and let S be a σ-progenerator. If* $\mathrm{Pic}_\sigma(S) = \mathrm{Pic}_\tau(S) = \mathrm{Pic}_\tau(S \otimes S) = 1$, *then* $\mathrm{Bcl}(S/R, \sigma, \tau) \cong H^1(S/R, Pu_{\sigma\tau})$.

Proof. Consider the following diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
1 & & 1 & & 1 & & 1 & & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Pic}_\sigma(R) & \to & \mathrm{Pic}_\tau(R) & \to & \mathrm{Bcl}(S/R, \sigma, \tau) & \to & \mathrm{Br}(S/R, \sigma) & \to & \mathrm{Br}(S/R, \tau) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Pic}_\sigma(R) & \to & \mathrm{Pic}_\tau(R) & \to & \mathrm{Bcl}(R, \sigma, \tau) & \to & \mathrm{Br}(R, \sigma) & \to & \mathrm{Br}(R, \tau) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & 1 & \to & \mathrm{Bcl}(S, \bar{\sigma}, \bar{\tau}) & \to & \mathrm{Br}(S, \bar{\sigma}) & \to & \mathrm{Br}(S, \bar{\tau})
\end{array}$$

The exactness of the second and the third rows follows from 1.1; a straightforward computation then leads to the exactness of the first row. Using arguments similar to the ones used in the proof of 1.2, we may find a map α: $H^1_\sigma(S/R, U_\tau) \to \mathrm{Pic}_\tau(R)$ such that $\mathrm{Im}(\alpha) = \mathrm{Im}(\mathrm{Pic}_\sigma(R) \to \mathrm{Pic}_\tau(R))$. Using the exact sequences $(*)$, [5, 3.2] and 1.2, we therefore have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
H^1_\sigma(S/R, U_\tau) & \to & \mathrm{Pic}(R, \tau) & \to & \mathrm{Bcl}(S/R, \sigma, \tau) & \to & \mathrm{Br}(S/R, \sigma) & \to & \mathrm{Br}(S/R, \tau) \\
= & & \cong & & & & \cong & & \cong \\
H^1_\sigma(S/R, U_\tau) & \to & H^1(S/R, U_\tau) & \to & H^1(S/R, Pu_{\sigma\tau}) & \to & H^2_\sigma(S/R, U_\tau) & \to & H^2(S/R, U_\tau)
\end{array}$$

The result then follows from the lemma of 5.

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